

## Functional Quantum Theory of Scattering Processes. II.

H. STUMPF

Institut für Theoretische Physik der Universität Tübingen

(Z. Naturforsch. 25 a, 796—803 [1970]; received 9 April 1970)

Dynamics of quantum field theory can be formulated by functional equations. To develop a complete functional quantum theory one has to describe the physical information by functional operations only. The most important physical information of elementary particle physics is the  $S$ -matrix. In this paper the functional  $S$ -matrix is constructed for relativistic spin 1/2 fermion scattering in nonlinear spinor theory with noncanonical relativistic Heisenberg quantization. With appropriate modifications the procedure runs on the same pattern as in the case of nonrelativistic potential scattering treated in I. Furthermore a calculational method for scattering functionals is proposed. In the appendices technical details are discussed.

In quantum field theory the dynamical behaviour of physical systems can be described by Schwinger functionals of the field operators and corresponding functional equations<sup>1-7</sup>. Using this representation of quantum dynamics it is tempting to develop a functional quantum theory in functional Hilbert space where the information of conventional quantum theory can be obtained by operations in functional Hilbert space only. The most important physical information in quantum field theory is given by the  $S$ -matrix. Therefore in functional quantum field theory one would like to construct a functional representation of the  $S$ -matrix. A first step to realize this program has been made by the introduction of an appropriate scalar product definition for Schwinger functionals<sup>8-10</sup> and by the derivation of a functional  $S$ -matrix for nonrelativistic potential scattering<sup>11</sup> using this scalar product definition. In this paper we consider the relativistic scattering of spin 1/2 particles. By an appropriate modification of the method used in<sup>11</sup> we construct the functional  $S$ -matrix for these processes. The construction does not depend on the special type of interaction but only on the Lorentz invariance of the theory, on the

validity of the so-called asymptotic condition for the field operators in its generalized form<sup>12</sup>, and on the spectral condition. Of special interest is the application to nonlinear spinor theory of elementary particles, regularized by noncanonical relativistic Heisenberg quantization<sup>13</sup> with indefinite metric. So we perform the entire procedure by assuming the nonlinear spinor equation to be the basic field equation. Then in the functional formulation of this theory also a calculational procedure can be given for the effective calculation of the corresponding scattering functionals. It should be noted that this functional  $S$ -matrix approach has sharply to be distinguished from functional causality and unitarity relations for the  $S$ -matrix<sup>7</sup> and from the current  $S$ -matrix representation by vacuum expectation values (v.e.v.)<sup>14</sup>. The latter one is not very suitable as no calculational method for v.e.v. exists and for indefinite metric the connection between the v.e.v. and the  $S$ -matrix is not clarified. The method given here is arranged especially to avoid these drawbacks, but of course further investigations are necessary to develop a complete functional  $S$ -matrix theory in the sense defined above.

### 1. Fundamentals

First we discuss the essentials of nonlinear spinor theory in its functional formulation. For details we refer to<sup>4,5,13,15</sup>. A real Hermitean field operator  $\Psi_\alpha(x)$  is assumed, describing the nonlinear spinor field

Sonderdruckanforderungen an Prof. Dr. H. STUMPF, D-7400 Tübingen, Wächterstr. 67.

<sup>1</sup> J. SCHWINGER, Proc. Nat. Acad. Sci. **37**, 452 [1951].

<sup>2</sup> J. SCHWINGER, Phys. Rev. **92**, 1283 [1953].

<sup>3</sup> K. SYMANZIK, Z. Naturforsch. **9a**, 809 [1954].

<sup>4</sup> H. RAMPACHER, H. STUMPF and F. WAGNER, Fortschr. Phys. **13**, 385 [1965].

<sup>5</sup> H. P. DÜRR and F. WAGNER, Nuovo Cim. X, **46**, 223 [1966].

<sup>6</sup> D. LURIE, Particles and Fields, Interscience Publ. New York 1968.

<sup>7</sup> J. RZEWUSKI, Field Theory, Vol. 2, Iliffe Books L.T.D., London 1969.

<sup>8</sup> H. STUMPF, Z. Naturforsch. **24a**, 188 [1969].

<sup>9</sup> H. STUMPF, Z. Naturforsch. **25a**, in press.

<sup>10</sup> R. WEBER, Thesis, University of Tübingen 1970.

<sup>11</sup> H. STUMPF, Z. Naturforsch. **24a**, 1022 [1969].

<sup>12</sup> R. HAAG, Recent developments in particle physics, Proc. Pacific Intern. Summer School in Physics, 1965.

<sup>13</sup> W. HEISENBERG, An Introduction to the Unified Theory of Elementary Particles, Wiley and Sons, London 1967.

<sup>14</sup> R. HAGEDORN, Fortschr. Phys., Sonderband 5 [1963].

<sup>15</sup> H. STUMPF, Z. Naturforsch., to be publ.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

in physical Hilbert space. The generating Schwinger functionals are defined by

$$\begin{aligned}\mathfrak{T}(j) &:= \langle a | T \exp \{ i \int \Psi_{\alpha}(x) j^{\alpha}(x) dx \} | b \rangle \\ &:= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle a | T \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | b \rangle j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) dx_1 \dots dx_n\end{aligned}\quad (1.1)$$

where  $\langle a |$  and  $| b \rangle$  are states of the system in physical Hilbert space and the sources  $j^{\alpha}(x)$  are elements of an operator algebra which is completed by functional derivatives  $\partial^{\alpha}(x)$ . For details of this algebra and the transformation properties of fields and sources we refer to <sup>9</sup>. By means of the procedure described in <sup>9</sup> the following functional equation for  $|\mathfrak{T}(j)\rangle$  can be derived<sup>15</sup>

$$\{D_{\alpha}^{\beta}(x) \partial_{\beta}(x) + V_{\alpha}^{\beta\gamma\delta}[\partial_{\beta}(x) \partial_{\gamma}(x) \partial_{\delta}(x) + 3 F_{\beta\gamma}(0) \partial_{\delta}(x)] - i \varrho_0 j_{\alpha}(x)\} |\mathfrak{T}(j)\rangle = 0 \quad (1.2)$$

where  $|\mathfrak{T}(j)\rangle := \mathfrak{T}(j) | \varphi_0 \rangle$  and  $|\varphi_0\rangle$  is the functional vacuum state<sup>9</sup>. In noncanonical quantized theory  $F_{\beta\gamma}(0)$  and  $\varrho_0$  are assumed to disappear and so we omit these quantities in the following. Equation (1.2) can be changed into an integral equation by applying the causal Green-Function  $D^{-1}(x)$  to (1.2). Additionally the resulting equation is averaged by a functional operator  $s^{\alpha}(x)$  as no special value of  $x$  is preferred. This gives

$$[\int s^{\alpha}(x) \partial_{\alpha}(x) dx + \int s^{\alpha}(x) G_{\alpha}^{\alpha}(x - x') V_{\alpha}^{\beta\gamma\delta} \partial_{\beta}(x') \partial_{\gamma}(x') \partial_{\delta}(x') dx dx'] |\mathfrak{T}(j)\rangle = 0. \quad (1.3)$$

For details of the averaging see<sup>15,16,17</sup>. If the states  $\langle a |$  and  $| b \rangle$  are pure or partly pure states with definite quantum numbers, then it can be shown that the functionals (1.1) satisfy subsidiary conditions<sup>18</sup>. These conditions serve to characterize the corresponding functionals in functional space. Specializing to  $\langle a | \equiv \langle 0 |$  i.e. the physical ground state and assuming for  $| b \rangle$  the quantum numbers  $p$  (four momentum),  $\mu$  (mass),  $s$  (total spin), and  $s_3$  (spin component in  $e_3$ -direction) we denote the corresponding functionals by  $|\mathfrak{T}_b(j)\rangle$  with  $b := (p, \mu, s, s_3)$  and obtain the subsidiary conditions

$$\begin{aligned}\mathfrak{P}_h |\mathfrak{T}_b(j)\rangle &= p_h |\mathfrak{T}_b(j)\rangle; & \mathfrak{P}^2 |\mathfrak{T}_b(j)\rangle &= \mu^2 |\mathfrak{T}_b(j)\rangle; \\ \mathfrak{G}_{\mu} \mathfrak{G}^{\mu} |\mathfrak{T}_b(j)\rangle &= s(s+1) |\mathfrak{T}_b(j)\rangle; & \mathfrak{S}_3 |\mathfrak{T}_b(j)\rangle &= s_3 |\mathfrak{T}_b(j)\rangle.\end{aligned}\quad (1.4)$$

For details see<sup>18</sup>. The subsidiary conditions (1.4) result only from the Lorentz form invariance of the theory. Conditions resulting from gauge groups etc. are not considered as they depend on the special model of interaction.

## 2. Asymptotic Functional States

According to (1.4) and <sup>18</sup> the functionals can be characterized by quantum numbers resulting from the corresponding states in physical Hilbert space. Therefore the functional quantum numbers have a well defined physical meaning namely the usual physical interpretation. The maximal set of them is denoted by  $\mathfrak{K} := k, m$  where  $m$  is the total mass of the particle (system) and  $k := p, s$  with  $p \equiv$  total four momentum,  $s \equiv$  total spin and additional quantum numbers. Formulating the theory in its functional version as given in sect. 1 these quantum numbers are the eigenvalues of stationary state functionals, which have to be calculated by functional methods<sup>15</sup>. We assume now that stationary solutions of functional theory exist describing spin 1/2 particles of different masses. Further we assume that the corresponding mass eigenvalues are a discrete, denumerable finite set, which we denote by  $\mu_1 \dots \mu_N$ . Of course all sets of quantum numbers which can be obtained by symmetry transformations from the set  $\{\mu\}$  are eigenvalues, too. Then in physical Hilbert space there exist dressed one particle states  $|p, \sigma_k, \mu_i\rangle$  with the same quantum numbers corresponding to the eigenfunctionals of the functional version. In this case the generalized asymptotic condition can be written in the sense of weak convergence<sup>12</sup>.

$$\lim_{t \rightarrow \infty} \Psi_{\alpha}(x) = \lim_{t \rightarrow \infty} \sum_{i=1}^N \sum_k [\int \langle p, \sigma_k, \mu_i | \Psi_{\alpha}(x) | 0 \rangle a^{+}(p, \sigma_k, \mu_i) dp + \int \langle 0 | \Psi_{\alpha}(x) | p, \sigma_k, \mu_i \rangle a(p, \sigma_k, \mu_i) dp] \quad (2.1)$$

<sup>16</sup> W. SCHULER and H. STUMPF, Z. Naturforsch. **22a**, 1842 [1967].

<sup>17</sup> W. SCHULER and H. STUMPF, Z. Naturforsch. **23a**, 902 [1968].

<sup>18</sup> H. STUMPF, H. G. MÄRTL and K. SCHEERER, Z. Naturforsch., to be published.

where  $a^+(p\sigma_k\mu_i)$  resp.  $a(p\sigma_k\mu_i)$  are the creation resp. destruction operators of these particles in the asymptotic free field description. Actually (2.1) is assumed to hold only for quasilocal field operators. For simplicity we extended it to local ones, as the  $S$ -matrix construction becomes very transparent in this version. If necessary the construction can be performed also for quasilocal operators, but this shall not be discussed in the following. Due to their transformation properties<sup>18</sup> the matrix elements can be evaluated to give

$$\langle 0 | \Psi_\alpha(x) | p\sigma_k\mu_i \rangle = a_i c_\alpha(p, \sigma_k\mu_i) e^{ipx} = a_i f_\alpha(x | p\sigma_k\mu_i) \quad (2.2)$$

where  $c_\alpha$  is an eigenvector for definite  $p\sigma_k\mu_i$  in spinor space. The only quantity not to be fixed by group theoretical considerations is  $a_i$ . As for free Hermitean fields  $\Psi_\alpha(x\mu_i)$  of mass  $\mu_i$  the decomposition

$$\Psi_\alpha(x\mu_i) = \sum_k \int [f_\alpha(x | p\sigma_k\mu_i) a^+(p\sigma_k\mu_i) + f_\alpha^\times(x | p\sigma_k\mu_i) a(p\sigma_k\mu_i)] dp \quad (2.3)$$

is valid, we introduce the auxiliary field

$$X_\alpha(x) := \sum_{i=1}^N \Psi_\alpha(x\mu_i) a_i \quad (2.4)$$

and may write (2.1) observing (2.2) and (2.3)

$$\lim_{t \rightarrow \infty} \Psi_\alpha(x) = \lim_{t \rightarrow \infty} X_\alpha(x). \quad (2.5)$$

For the following it is useful to define corresponding auxiliary functionals by

$$\mathfrak{T}(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) := \langle 0 | T \exp \{ i \int X_\alpha(x) j^\alpha(x) dx \} | \mathfrak{R}_1 \dots \mathfrak{R}_n \rangle \quad (2.6)$$

where the corresponding states in physical Hilbert space are given by

$$| \mathfrak{R}_1 \dots \mathfrak{R}_n \rangle := a^+(p_1 s_1 m_1) \dots a^+(p_n s_n m_n) | 0 \rangle \quad (2.7)$$

with maximal sets of quantum numbers  $s_r \in \{\sigma_k\}$  and  $m_r \in \{\mu_i\}$ . As (2.4) is a linear combination of free field operators the Wick rule can be applied to give

$$\mathfrak{T}(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) = \exp \{ \frac{1}{2} \int j^\alpha(x) F_{\alpha\beta}(x, y) j^\beta(y) dx dy \} \Phi(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) \quad (2.8)$$

with

$$F_{\alpha\beta}(x, y) := \langle 0 | T X_\alpha(x) X_\beta(y) | 0 \rangle \quad (2.9)$$

and

$$\begin{aligned} \Phi(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) &:= \frac{1}{n!} \int \langle 0 | N X_{\alpha_1}(x_1) \dots X_{\alpha_n}(x_n) | \mathfrak{R}_1 \dots \mathfrak{R}_n \rangle j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) dx_1 \dots dx_n \\ &= \frac{1}{n!} p \sum_{v_1, \dots, v_n} (-1)^p \int f_{\alpha_1}(x_1 | \mathfrak{R}_{v_1}) \dots f_{\alpha_n}(x_n | \mathfrak{R}_{v_n}) a_{v_1} \dots a_{v_n} j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) dx_1 \dots dx_n. \end{aligned} \quad (2.10)$$

According to<sup>9</sup>; for the functionals (2.6) a functional scalar product can be defined by

$$\langle \overline{\mathfrak{T}(j, \mathfrak{R}'_1 \dots \mathfrak{R}'_m)} | \mathfrak{T}(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) \rangle_{w(\vartheta)} := \langle \exp \{ -\frac{1}{2} j(F + G_\vartheta) j \} \mathfrak{T}(j) | \exp \{ -\frac{1}{2} j(F + G_\vartheta) j \mathfrak{T}(j) \} \rangle. \quad (2.11)$$

We choose  $G_\vartheta$  to be a generalization of Ref.<sup>9</sup>, (3.5):

$$G_\vartheta(x x') := i({}_5G^\lambda G) n_\lambda f(n^\lambda x_\lambda - \vartheta) \delta(x - x') \quad (2.12)$$

where  ${}_5G^\lambda G$  are elements of the Hermitean Dirac algebra,  $n_\lambda$  is a timelike four vector and  $f(z)$  a function of compact support  $-z_0 \leq z \leq z_0$ ,  $z_0 = |z_0|$  finite with  $\int f(z) dz = \text{const}$  and  $|f(z)|$  bounded in the total range. Then we get the orthogonality relations

$$\lim_{\vartheta \rightarrow \infty} \langle \overline{\mathfrak{T}(j, \mathfrak{R}'_1 \dots \mathfrak{R}'_m)} | \mathfrak{T}(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) \rangle_{w(\vartheta)} = \frac{1}{n!} \delta_{nm} p \sum_{\lambda_1 \dots \lambda_n} (-1)^p \delta_{\mathfrak{R}'_1 \mathfrak{R}_{\lambda_1}} \dots \delta_{\mathfrak{R}'_m \mathfrak{R}_{\lambda_m}} a_1^2 \dots a_n^2. \quad (2.13)$$

For details of the proof see App. I.

### 3. S-Matrix Construction

To construct the  $S$ -matrix an extension of the definition (1.1) is required. Like in<sup>11</sup> we define the advanced and retarded Schwinger functionals for Schrödinger states by

$$\mathfrak{T}^{(\pm)}(j, \vartheta) := \langle 0 | T \exp \{ i \int \Psi_{\alpha}(x) j^{\alpha}(x) dx \} | a^{(\pm)}(\vartheta) \rangle \quad (3.1)$$

where the states  $|a^{(\pm)}(\vartheta)\rangle$  are the corresponding advanced resp. retarded Schrödinger states for the time  $\vartheta$ . These states can be characterized completely by the sets of quantum numbers (2.7) of the ingoing resp. outgoing free particles. Putting for brevity  $\mathfrak{N} := \mathfrak{N}_1 \dots \mathfrak{N}_n$  we may write  $|a_{\mathfrak{N}}^{(\pm)}(\vartheta)\rangle$  for a definite scattering state. To introduce a scalar product for scattering functionals we define the weighted scattering functionals in the meaning of (2.11) by

$$\exp \{ -\frac{1}{2} j(F + G_{\vartheta}) j \} | \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j, \vartheta) \rangle := \exp \{ -\frac{1}{2} j(F + G_{\vartheta}) j \} \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j, \vartheta) | \varphi_0 \rangle. \quad (3.2)$$

Then the functional scalar product reads

$$\langle \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j, \vartheta) | \mathfrak{T}_{\mathfrak{N}'}^{(\pm)}(j, \vartheta) \rangle_{w(\vartheta)} := \langle \exp \{ -\frac{1}{2} j(F + G_{\vartheta}) j \} \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j, \vartheta) | \exp \{ -\frac{1}{2} j(F + G_{\vartheta}) j \} \mathfrak{T}_{\mathfrak{N}'}^{(\pm)}(j, \vartheta) \rangle. \quad (3.3)$$

By means of these definitions the following statement can be proven:

The scattering matrix of spin 1/2 fermions with the initial state  $\mathfrak{N}$  and the final state  $\mathfrak{N}'$  is given by

$$S_{\mathfrak{N}'\mathfrak{N}} = \langle \mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, 0) | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, 0) \rangle_{w(0)} a_1^{-2} \dots a_n^{-2} \quad (3.4)$$

i.e. the  $S$ -matrix construction can be reduced to the scalar product of Schwinger functionals for Heisenberg states defined in (1.1).

*Proof:* We introduce the formally Wick ordered product by

$$\begin{aligned} \Phi_{\mathfrak{N}}^{(\pm)}(j, \vartheta) &:= \exp \{ -\frac{1}{2} j F j \} \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j, \vartheta) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \langle 0 | N \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | a_{\mathfrak{N}}^{(\pm)}(\vartheta) \rangle j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) dx_1 \dots dx_n \end{aligned} \quad (3.5)$$

with

$$\begin{aligned} \langle 0 | N \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | a_{\mathfrak{N}}^{(\pm)}(\vartheta) \rangle \\ := \sum_{\lambda_1 \dots \lambda_n} \sum_{n=0}^{[n/2]} \frac{i^n}{2^n r! (n-2r)!} F(x_{\lambda_1} x_{\lambda_2}) \dots F(x_{\lambda_{r-1}} x_{\lambda_r}) \langle 0 | T \Psi_{\alpha_{\lambda_{r+1}}}(x_{\lambda_{r+1}}) \dots \Psi_{\alpha_{\lambda_n}}(x_{\lambda_n}) | a_{\mathfrak{N}}^{(\pm)}(\vartheta) \rangle \end{aligned} \quad (3.6)$$

and we define the weighted Dyson functionals by

$$| D_n(x_1 \dots x_n, \vartheta) \rangle := \frac{1}{n!} j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) \exp \{ -\frac{1}{2} j G_{\vartheta} j \} | \varphi_0 \rangle. \quad (3.7)$$

Then we have

$$\begin{aligned} \langle \mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, \vartheta) | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, \vartheta) \rangle_{w(\vartheta)} &= \sum \int \langle 0 | N \overline{\Psi}_{\beta'_1}(x'_1) \dots \overline{\Psi}_{\beta'_n}(x'_n) | a_{\mathfrak{N}'}^{(+)}(\vartheta) \rangle \langle 0 | N \Psi_{\beta_1}(x_1) \dots \Psi_{\beta_m}(x_m) | a_{\mathfrak{N}}^{(-)}(\vartheta) \rangle \times \\ &\quad \times \langle D_n(x'_1 \dots x'_n, \vartheta) | D_m(x_1 \dots x_m, \vartheta) \rangle. \end{aligned} \quad (3.8)$$

Due to the Lorentz invariance of (2.9) and (2.12) the scalar product is Lorentz invariant, too, and (3.8) may be evaluated in a suitable frame of reference. This is the rest frame with  $n := (1, 0, 0, 0)$ . Then the procedure performed in <sup>9</sup> sect. 3 can be applied to (3.8) giving

$$\begin{aligned} \langle \mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, \vartheta) | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, \vartheta) \rangle_{w(\vartheta)} &= \sum_{n,m} \int \langle 0 | N \overline{\Psi}_{\beta'_1}(x'_1) \dots \overline{\Psi}_{\beta'_n}(x'_n) | a_{\mathfrak{N}'}^{(+)}(\vartheta) \rangle \langle 0 | N \Psi_{\beta_1}(x_1) \dots \Psi_{\beta_m}(x_m) | a_{\mathfrak{N}}^{(-)}(\vartheta) \rangle \\ &\times \sum_{\varrho=1}^{(n-2\varrho)/2} 2^{(2\varrho-n+m/2)} \frac{(-1)^{(n+m/2)}}{\left(\frac{n-2\varrho}{2}\right)! \left(\frac{m-2\varrho}{2}\right)!} \frac{1}{(2\varrho)!^2} {}^0G^{\beta'_1 \beta_1} \delta(x'_1 - x_1) b_1(t'_1 - \vartheta) \dots {}^0G^{\beta'_{2\varrho} \beta_{2\varrho}} \delta(x'_{2\varrho} - x_{2\varrho}) b_1(t'_{2\varrho} - \vartheta) \\ &\times Z^{\beta'_{2\varrho+1} \beta_{2\varrho+2}} \delta(x'_{2\varrho+1} - x'_{2\varrho+2}) b_2(t'_{2\varrho+2} - \vartheta) \dots Z^{\beta'_{n-1} \beta'_n} \delta(x'_{n-1} - x'_n) b_2(t'_n - \vartheta) \\ &\times Z^{\beta_{2\varrho+1} \beta_{2\varrho+2}} \delta(x_{2\varrho+1} - x_{2\varrho+2}) b_2(t_{2\varrho+2} - \vartheta) \dots Z^{\beta_{m-1} \beta_m} \delta(x_{m-1} - x_m) b_2(t_m - \vartheta) \cdot dx'_1 \dots dx'_n dx_1 \dots dx_m. \end{aligned} \quad (3.9)$$

Now for  $\vartheta \rightarrow \infty$  the following limes relations are valid:

$$\lim_{\substack{t_1 \rightarrow \infty \\ t_n \rightarrow \infty}} \langle 0 | N \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | a \rangle = \lim_{\substack{t_1 \rightarrow \infty \\ t_n \rightarrow \infty}} \langle 0 | N X_{\alpha_1}(x_1) \dots X_{\alpha_n}(x_n) | a \rangle \quad (3.10)$$

and

$$\lim_{\vartheta \rightarrow \infty} |a_{\mathfrak{N}'}^{(+)}(\vartheta)\rangle = \sum S_{\mathfrak{N}'\mathfrak{N}''} |\mathfrak{N}''\rangle, \quad \lim_{\vartheta \rightarrow \infty} |a_{\mathfrak{N}}^{(-)}(\vartheta)\rangle = |\mathfrak{N}\rangle. \quad (3.11, 3.12)$$

Observing  $b_1(t - \vartheta)$  and  $b_2(t - \vartheta)$  are functions of compact support  $-z_0 + \vartheta \leq t \leq z_0 + \vartheta$  the integrations in (3.9) run only over this interval and we are allowed to substitute (3.10), (3.11), (3.12) into (3.9) for  $\lim \vartheta \rightarrow \infty$ . Performing this and observing (2.13) we obtain

$$\lim_{\vartheta \rightarrow \infty} \langle \overline{\mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, \vartheta)} | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, \vartheta) \rangle_{w(\vartheta)} = \lim_{\vartheta \rightarrow \infty} \sum_{\mathfrak{N}''} S_{\mathfrak{N}'\mathfrak{N}''} \langle \overline{\mathfrak{T}(j, \mathfrak{N}'')} | \mathfrak{T}(j, \mathfrak{N}) \rangle_{w(\vartheta)} = S_{\mathfrak{N}'_1 \dots \mathfrak{N}'_n, \mathfrak{N}_1 \dots \mathfrak{N}_n} a_1^2 \dots a_n^2. \quad (3.13)$$

Therefore

$$S_{\mathfrak{N}'\mathfrak{N}} = \lim_{\vartheta \rightarrow \infty} \langle \overline{\mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, \vartheta)} | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, \vartheta) \rangle_{w(\vartheta)} a_1^{-2} \dots a_n^{-2}. \quad (3.14)$$

In App. II the translational invariance of the scalar product is proven. Therefore

$$\langle \overline{\mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, 0)} | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, 0) \rangle_{w(0)} = \langle \overline{\mathfrak{T}_{\mathfrak{N}'}^{(+)}(j, \vartheta)} | \mathfrak{T}_{\mathfrak{N}}^{(-)}(j, \vartheta) \rangle_{w(\vartheta)} \quad (3.15)$$

holds for any  $\vartheta$ . From the combination of this with (3.14) (3.4) follows q.e.d.

#### 4. Normalization of Stationary State Functionals

In the last section the actual calculation of the constants  $a_i$  defined by (2.2) remained open. In this section we show that their calculation is tightly connected with the correct normalization of stationary state functionals. These functionals are defined for „dressed„ one particle states  $|\mathfrak{p}, \sigma_k, \mu_i\rangle_n$  with  $n \langle \mathfrak{p}, \sigma_k, \mu_i | \mathfrak{p}, \sigma_k, \mu_i \rangle_n = 1$  where the lower index  $n$  indicates that these one particle states are assumed to be normalized in physical Hilbert space. The corresponding functionals for the Schrödinger states are defined by ( $\mathfrak{N}_i := \mathfrak{p} \sigma_k \mu_i$ )

$$t_{\mathfrak{N}_i}(j, \vartheta) := \langle 0 | T \exp\{i \int \Psi_{\alpha}(x) j^{\alpha}(x) dx\} | \mathfrak{N}_i(\vartheta) \rangle_n. \quad (4.1)$$

For the actual calculation of these functionals it is important to note that they cannot be obtained by the solution of the functional Eqs. (1.3), (1.4) alone. Because these equations are homogeneous for stationary states the absolute value of the functional solution remains open. This means that by the eigenvalue equation only a functional

$$\mathfrak{T}_{\mathfrak{N}_i}(j, \vartheta) = \langle 0 | T \exp\{i \int \Psi_{\alpha}(x) j^{\alpha}(x) dx\} | \mathfrak{N}_i(\vartheta) \rangle \quad (4.2)$$

can be calculated where  $|\mathfrak{N}_i\rangle$  is not properly normalized. To fix the absolute value of  $\mathfrak{T}_{\mathfrak{N}_i}$  at all we may impose an arbitrary normalization condition. As we shall see later it is convenient to require

$$\langle \overline{\mathfrak{T}_{\mathfrak{N}_i}(j, \vartheta)} | \mathfrak{T}_{\mathfrak{N}_i}(j, \vartheta) \rangle_{w(\vartheta)} = 1 \quad (4.3)$$

with the definition (3.3) of the scalar product. Assuming the existence of (4.3) this condition fixes the absolute value of  $\mathfrak{T}_{\mathfrak{N}_i}$  and therefore causes a certain norm value of  $|\mathfrak{N}_i(\vartheta)\rangle$  in physical Hilbert space. Denoting this value by  $|\lambda_i|^2$  we obtain

$$|\mathfrak{N}_i(\vartheta)\rangle = \lambda_i |\mathfrak{N}_i(\vartheta)\rangle_n \quad (4.4)$$

and therefore

$$|\mathfrak{T}_{\mathfrak{N}_i}(j, \vartheta)\rangle = \lambda_i |\mathfrak{T}_{\mathfrak{N}_i}(j, \vartheta)\rangle_n. \quad (4.5)$$

By (4.3), (4.4), (4.1), (4.2) then follows

$$\langle \overline{t_{\mathfrak{N}_i}(j, \vartheta)} | t_{\mathfrak{N}_i}(j, \vartheta) \rangle_{w(\vartheta)} = 1/|\lambda_i|^2. \quad (4.6)$$



On the other hand we may perform the limes procedure  $\vartheta \rightarrow \infty$  on the norm of  $t_{\mathfrak{R}_i}(j, \vartheta)$ . Due to the properties of the weighting function  $G_\vartheta$  this means a substitution of  $\Psi$  by  $X$  in the norm expression. Observing further that the dressed one particle state  $|\mathfrak{R}_i(\vartheta)\rangle$  may be expressed also in the free field representation by the application of a creation operator  $a^+(\mathfrak{R}_i)$  on  $|0\rangle$  we get for the limes procedure

$$\lim_{\vartheta \rightarrow \infty} \langle \overline{t_{\mathfrak{R}_i}(j, \vartheta)} | t_{\mathfrak{R}_i}(j, \vartheta) \rangle_{w(\vartheta)} = \lim_{\vartheta \rightarrow \infty} \langle \overline{\mathfrak{T}(j, \mathfrak{R}_i)} | \mathfrak{T}(j, \mathfrak{R}_i) \rangle_{w(\vartheta)} = a_i \quad (4.7)$$

with the definitions of Section 2. As the norm value of  $t_{\mathfrak{R}_i}$  is independent of  $\vartheta$  according to App. II, combination of (4.7) and (4.6) leads to

$$|\lambda_i|^2 a_i = 1. \quad (4.8)$$

To obtain a second equation we consider the matrix element

$$\partial_\alpha(x) \mathfrak{T}_{\mathfrak{R}_i}(j, 0) | \varphi_0 \rangle_{j=0} = \langle 0 | \Psi_\alpha(x) | \mathfrak{R}_i(0) \rangle = g_\alpha(\mathfrak{R}_i) e^{ipx}. \quad (4.9)$$

Due to (4.3) the numerical value of  $g_\alpha(\mathfrak{R}_i)$  is completely fixed. Further by (2.2) follows

$$\langle 0 | \Psi_\alpha(x) | \mathfrak{R}_i(0) \rangle = \lambda_i \langle 0 | \Psi_\alpha(x) | \mathfrak{R}_i(0) \rangle_n = \lambda_i a_i f_\alpha(\mathfrak{R}_i) e^{ipx}. \quad (4.10)$$

Dividing (4.9) by (4.10) gives

$$g_\alpha(\mathfrak{R}_i) = \lambda_i a_i f_\alpha(\mathfrak{R}_i). \quad (4.11)$$

By Lorentz invariance requirements  $a_i$  may not depend on the special frame of reference. Therefore for its calculation we use the rest frame. This gives by combining (4.8) and (4.11)

$$a_i = \left( \frac{g_\alpha(\mu_i)}{f_\alpha(\mu_i)} \right)^2 \quad (4.12)$$

where  $f_\alpha(\mu_i)$  is an orthonormalized solution of a spin 1/2 particle with mass  $\mu_i$ . Therefore assuming (4.3) to be valid  $a_i$  can be calculated by (4.12). (4.12) contains also the desired proof that in the asymptotic condition (2.1) only those particles appear which are eigenstates of the functional equation. This follows immediately from (4.9). Only for eigenstates  $g_\alpha(\mu_i) \neq 0$  is possible as for mass values being no eigenvalues of the system only  $|\mathfrak{T}(j)\rangle \equiv 0$  satisfies the eigenvalue equation. Therefore  $a_i \neq 0$  can hold only for mass eigenvalues q.e.d.

## 5. Calculation of Scattering Functionals

To complete functional quantum theory of scattering also a calculational method for the explicit construction of scattering functionals is required. This is provided by the least square method proposed in <sup>11</sup> applied successfully already to Bethe-Salpeter problems <sup>19</sup>. Its formulation for relativistic functional equations in the case of stationary state functionals is given in <sup>15</sup>. To apply this method to functional scattering problems one has to know additionally the appropriate boundary or initial conditions which have to be satisfied by the scattering functionals themselves. We assume that this condition is given by the decomposition

$$\mathfrak{T}_{\mathfrak{R}}^{(\pm)}(j, 0) = \mathfrak{T}(j, \mathfrak{R}) + \mathfrak{D}_{\mathfrak{R}}^{(\pm)}(j, 0) \quad (5.1)$$

where  $\mathfrak{T}(j, \mathfrak{R})$  is the free particle functional (2.7). Performing the variational procedure only  $\mathfrak{D}_{\mathfrak{R}}^{(\pm)}$  may be varied, whereas  $\mathfrak{T}(j, \mathfrak{R})$  remains fixed characterizing the ingoing resp. outgoing free particle configuration  $\mathfrak{R}$ . In the case of potential scattering this decomposition could be proven rigorously <sup>11</sup>. For the present case we confine ourselves to plausible arguments, which later might be extended to a rigorous proof. Using the definitions (3.1) (2.5) we may write

$$\mathfrak{T}_{\mathfrak{R}}^{(\pm)}(j, 0) = \langle 0 | T \exp \{ i \int X_\alpha(x) j^\alpha(x) dx + i \int [\Psi_\alpha(x) - X_\alpha(x)] j^\alpha(x) dx \} a_{\mathfrak{R}}^{(\pm)}(0) \rangle. \quad (5.2)$$

In physical Hilbert space the decomposition is valid

$$| a_{\mathfrak{R}}^{(\pm)}(0) \rangle = | \mathfrak{R}(0) \rangle + \mathbf{K} | \mathfrak{R}(0) \rangle \quad (5.3)$$

<sup>19</sup> K. LADANYI, NUOVO Cim. X. **61**, 173 (1969).

where  $\mathbf{K}$  is an operator which needs not to be specified further. Substituting (5.3) into (5.2) and expanding the exponential we arrive just at (5.1) q.e.d. For trial functionals we use the truncated functionals as defined in <sup>15</sup>. All other steps run on the same pattern.

### Appendix I

In this appendix we prove the functional orthonormalization of the asymptotic free functional states which are described by the auxiliary functionals (2.6). Since in (2.6) only free fields occur, the functional orthonormalization can be reduced to ordinary scalar products of Dirac spinors <sup>9</sup>. For these the following theorem holds:

If  $\psi_\alpha(x|\mathfrak{R}_i)$  are ordinary Dirac wave functions with the quantum numbers  $\mathfrak{R}_i, i = 1, 2, \mathfrak{R}_i := p_i, s_i, m_i$  and if  $f(z)$  is a function of compact support  $-z_0 \leq z \leq z_0, z_0 = |z_0|$  finite with  $\int f(z) dz = 1$  and  $|f(z)|$  bounded in the total range, then

$$\lim_{\vartheta \rightarrow \infty} \int \bar{\psi}(x_1|\mathfrak{R}_1) \gamma^\lambda n_\lambda \delta(x_1 - x_2) f(n^\mu x_{1\mu} - \vartheta) \psi(x_2|\mathfrak{R}_2) dx_1 dx_2 = \delta_{\mathfrak{R}_1 \mathfrak{R}_2} \quad (\text{I.1})$$

is valid for timelike  $n_\lambda$ .

*Proof:* As  $n_\lambda$  is timelike a Lorentz transformation  $a_\lambda^{\rho}$  exists with  $n'_\lambda = a_\lambda^{\rho} n_\rho$  and  $n'_\lambda = (1, 0, 0, 0)$ . The spinorial wave functions  $\psi_\alpha(x|\mathfrak{R}) := \psi_\alpha(\mathfrak{R}) e^{-ixp}$  transform for  $x'_\lambda = a_\lambda^{\rho} x_\rho$  according to the law

$$\psi'_\alpha = S_{\alpha\beta}(a) \psi_\beta(\mathfrak{R}) e^{-ix'a-1p} = \psi_\alpha(\mathfrak{R}') e^{-ix'p'}. \quad (\text{I.2})$$

Therefore the following identities are valid

$$\psi_\alpha(x|\mathfrak{R}) = S_{\alpha\beta}^{-1}(a) \psi_\beta(ax|\mathfrak{R}'), \quad \bar{\psi}_\alpha(x|\mathfrak{R}) = \bar{\psi}_\beta(ax|\mathfrak{R}') S_{\beta\alpha}(a). \quad (\text{I.3})$$

Substituting (I.3) into (I.1) we obtain

$$\int \bar{\psi}(x|\mathfrak{R}_1) \gamma^\lambda n_\lambda f(n^\mu x_\mu - \vartheta) \psi(x|\mathfrak{R}_2) dx = \int \bar{\psi}(ax|\mathfrak{R}'_1) S \gamma^\lambda S^{-1} n_\lambda f(n^\mu x_\mu - \vartheta) \psi(ax|\mathfrak{R}'_2) dx. \quad (\text{I.4})$$

Observing  $S(a) \gamma^\lambda S^{-1}(a) = S^{-1}(a^{-1}) \gamma^\lambda S(a^{-1}) = a_\lambda^{\rho} \gamma^\rho$  (I.4) changes into

$$\int \bar{\psi}(ax|\mathfrak{R}'_1) n'_\nu \gamma^\nu f(n^\mu x_\mu - \vartheta) \psi(ax|\mathfrak{R}'_2) dx \quad (\text{I.5})$$

and with  $\xi_\mu = a_\mu^{\rho} x_\rho$  we obtain by observing the value of  $n'_\lambda$

$$\int \bar{\psi}(x|\mathfrak{R}_1) \gamma^\lambda n_\lambda f(n^\mu x_\mu - \vartheta) \psi(x|\mathfrak{R}_2) dx = \int \bar{\psi}(\xi|\mathfrak{R}'_1) \gamma_0 f(\xi_0 - \vartheta) \psi(\xi|\mathfrak{R}'_2) d\xi. \quad (\text{I.6})$$

Now the right side of (I.6) can be evaluated to give

$$\int \bar{\psi}(\xi|\mathfrak{R}'_1) \gamma_0 f(\xi_0 - \vartheta) \psi(\xi|\mathfrak{R}'_2) d\xi = \delta(p'_1 - p'_2) \delta_{s'_1 s'_2} \int \exp\{i(\omega'_1 - \omega'_2) t\} f(t - \vartheta) dt \quad (\text{I.7})$$

with  $\omega'_i = (p_i'^2 + m_i'^2)^{1/2}$ . Applying the theory of distributions we obtain

$$\lim_{\vartheta \rightarrow \infty} \delta_{s'_1 s'_2} \delta(p'_1 - p'_2) \int \exp\{i(\omega'_1 - \omega'_2) t\} f(t - \vartheta) dt = \delta_{m'_1 m'_2} \delta(p'_1 - p'_2) \delta_{s'_1 s'_2} \quad (\text{I.8})$$

and therefore

$$\lim_{\vartheta \rightarrow \infty} \int \bar{\psi}(x|\mathfrak{R}_1) \gamma^\lambda n_\lambda f(n^\mu x_\mu - \vartheta) \psi(x|\mathfrak{R}_2) dx = \delta(p'_1 - p'_2) \delta_{s'_1 s'_2} \delta_{m'_1 m'_2}. \quad (\text{I.9})$$

Now we have  $\mathfrak{R}'_i := p'_i s'_i m'_i$  and  $\mathfrak{R}_i := p_i s_i m_i$ . Due to the relativistic invariance it is  $m_\alpha = m'_\alpha$  and therefore

$$\lim_{\vartheta \rightarrow \infty} \int \bar{\psi}(x|\mathfrak{R}_1) \gamma^\lambda n_\lambda f(n^\mu x_\mu - \vartheta) \psi(x|\mathfrak{R}_2) dx = \delta(p'_1 - p'_2) \delta_{s'_1 s'_2} \delta_{m_1 m_2}. \quad (\text{I.10})$$

But for  $m_1 = m_2$  (I.10) can be calculated explicitly (see Ref. <sup>9</sup>, (I.33)) to give finally

$$\lim_{\vartheta \rightarrow \infty} \int \bar{\psi}(x|\mathfrak{R}_1) \gamma^\lambda n_\lambda f(n^\mu x_\mu - \vartheta) \psi(x|\mathfrak{R}_2) dx = \delta(p_1 - p_2) \delta_{s_1 s_2} \delta_{m_1 m_2} \quad \text{q.e.d.} \quad (\text{I.11})$$

The further development runs in complete analogy to (see Ref. <sup>9</sup>, section 3). Especially we obtain in the Hermitean representation by evaluation in the rest system

$$\begin{aligned}
 & \lim_{\vartheta \rightarrow \infty} \langle \overline{\mathfrak{T}(j, \mathfrak{R}'_1 \dots \mathfrak{R}'_n)} | \mathfrak{T}(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) \rangle_{w(\vartheta)} \\
 &= \lim_{\vartheta \rightarrow \infty} \int p' \sum_{\lambda'_1 \dots \lambda'_n} (-1)^{p'} f_{\beta'_1}(x'_1 | \mathfrak{R}'_{\lambda'_1}) \dots f_{\beta'_n}(x'_n | \mathfrak{R}'_{\lambda'_n})^p \sum_{\lambda_1 \dots \lambda_m} (-1)^p f_{\beta_1}(x_1 | \mathfrak{R}_{\lambda_1}) \dots f_{\beta_m}(x_m | \mathfrak{R}_{\lambda_m}) a_{\lambda'_1} \dots a_{\lambda'_n} a_{\lambda_1} \dots a_{\lambda_m} \\
 & \quad \times \sum_{\varrho=1}^{(n-2\varrho)/2} 2^{(n+m)/2} \frac{(-1)^{(n+m)/2}}{\left(\frac{n-2\varrho}{2}\right)! \left(\frac{m-2\varrho}{2}\right)!} {}_0G^{\beta'_1 \beta_1} \delta(x'_1 - x_1) b_1(t'_1 - \vartheta) \dots {}_0G^{\beta'_{2\varrho} \beta_{2\varrho}} \delta(x'_{2\varrho} - x_{2\varrho}) b_1(t'_{2\varrho} - \vartheta) \\
 & \quad \times Z^{\beta'_{2\varrho+1} \beta_{2\varrho+2}} \delta(x'_{2\varrho+1} - x'_{2\varrho+2}) b_2(t'_{2\varrho+2} - \vartheta) \dots Z^{\beta'_{n-1} \beta_n} \delta(x'_{n-1} - x'_n) b_2(t'_n - \vartheta) \\
 & \quad \times Z^{\beta_{2\varrho+1} \beta_{2\varrho+2}} \delta(x_{2\varrho+1} - x_{2\varrho+2}) b_2(t_{2\varrho+2} - \vartheta) \dots Z^{\beta_{m-1} \beta_m} \delta(x_{m-1} - x_m) b_2(t_m - \vartheta) \\
 & \quad \cdot dx \dots dx_1 \dots dx_m.
 \end{aligned} \tag{I.12}$$

Dividing (I.11) into real and imaginary parts like in <sup>9</sup> it follows from (I.11) in the rest system

$$\begin{aligned}
 \delta_{\mathfrak{N}\mathfrak{N}} &= \lim_{\vartheta \rightarrow \infty} \int \psi_{\mathfrak{N}}^x(x | \mathfrak{N}) f(t - \vartheta) \psi_{\mathfrak{N}}(x | \mathfrak{N}') dx \\
 &= \lim_{\vartheta \rightarrow \infty} [\int [f_{1\alpha} f'_{1\alpha} + f_{2\alpha} f'_{2\alpha}] f(t - \vartheta) dx + i \int [f_{1\alpha} f'_{2\alpha} - f_{2\alpha} f'_{1\alpha}] f(t - \vartheta) dx]
 \end{aligned} \tag{I.13}$$

with  $\psi_{\alpha} = f_{1\alpha} + i f_{2\alpha}$  and as  $f_1, f_2$  are completely real

$$\lim_{\vartheta \rightarrow \infty} \int [f_{1\alpha} f'_{1\alpha} + f_{2\alpha} f'_{2\alpha}] f(t - \vartheta) dx = \delta_{\mathfrak{N}\mathfrak{N}'}, \quad \lim_{\vartheta \rightarrow \infty} \int [f_{1\alpha} f'_{2\alpha} - f_{2\alpha} f'_{1\alpha}] f(t - \vartheta) dx = 0. \tag{I.14}$$

As

$$\begin{aligned}
 \int f_{\beta'}(x | \mathfrak{N}) {}_0G^{\beta' \beta} f_{\beta}(x | \mathfrak{N}') b_1(t - \vartheta) dx &= \int [f_{1\alpha} f'_{1\alpha} + f_{2\alpha} f'_{2\alpha}] b_1(t - \vartheta) dx, \\
 \int f_{\beta'}(x | \mathfrak{N}) Z^{\beta' \beta} f_{\beta}(x | \mathfrak{N}') b_2(t - \vartheta) dx &= \int [f_{1\alpha} f'_{2\alpha} - f_{2\alpha} f'_{1\alpha}] b_2(t - \vartheta) dx
 \end{aligned} \tag{I.15}$$

it follows

$$\lim_{\vartheta \rightarrow \infty} \langle \overline{\mathfrak{T}(j, \mathfrak{N} \dots \mathfrak{N})} | \mathfrak{T}(j, \mathfrak{N}_1 \dots \mathfrak{N}_m) \rangle_{w(\vartheta)} = \delta_{nm} \frac{1}{n!} p \sum_{\lambda_1 \dots \lambda_n} (-1)^p \delta_{\mathfrak{N}'_1 \mathfrak{N}_{\lambda_1}} \dots \delta_{\mathfrak{N}'_n \mathfrak{N}_{\lambda_n}} a_1^2 \dots a_n^2. \tag{I.16}$$

## Appendix II

In this appendix we prove the invariance of the functional scalar product under time translations i.e. formula (3.15). The general definition of the weighted functionals is with  $W_{\vartheta} := F + G_{\vartheta}$

$$\begin{aligned}
 \exp\{-\tfrac{1}{2} j W_{\vartheta} j\} | \mathfrak{T}_a(j, \vartheta) : &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T \Psi(x_1) \dots \Psi(x_n) | a(\vartheta) \rangle \\
 &\quad \times \exp\{-\tfrac{1}{2} \int j(\xi) W_{\vartheta}(\xi, \xi') j(\xi') d\xi d\xi'\} j(x_1) \dots j(x_n) | \varphi_0 \rangle dx_1 \dots dx_n.
 \end{aligned} \tag{II.1}$$

The Hermitean conjugate with respect to the functional space is

$$\begin{aligned}
 (\exp\{-\tfrac{1}{2} j W_{\vartheta} j\} | \mathfrak{T}_a(j, \vartheta) \rangle)^+ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T \Psi(x_1) \dots \Psi(x_n) | a(\vartheta) \rangle \langle \varphi_0 | j^+(x_n) \dots j^+(x_1) \\
 &\quad \times \exp\{-\tfrac{1}{2} \int j^+(\xi) W_{\vartheta}(\xi, \xi') j^+(\xi') d\xi d\xi'\} dx_1 \dots dx_n.
 \end{aligned} \tag{II.2}$$

Observing according to <sup>1</sup>  $\partial(x) = j^+(x)$  the adjoint functional of (II.1) is defined by

$$\begin{aligned}
 \langle \exp\{-\tfrac{1}{2} j W_{\vartheta} j\} | \mathfrak{T}_a(j, \vartheta) : &= \sum_{n=0}^{\infty} \frac{(-i)^n}{i^n} \int \langle 0 | T \Psi(x_1) \dots \Psi(x_n) | a(\vartheta) \rangle \langle \varphi_0 | \partial(x_1) \dots \partial(x_n) \\
 &\quad \times \exp\{-\tfrac{1}{2} \int \partial(\xi) \overline{W_{\vartheta}(\xi, \xi')} \partial(\xi') d\xi d\xi'\} dx_1 \dots dx_n
 \end{aligned} \tag{II.3}$$

where the bar denotes the ordinary spinorial adjoint of quantum mechanics. Then the scalar product is defined by



$$\begin{aligned} \langle \overline{\mathfrak{T}_a(j, \vartheta)} | \mathfrak{T}_b(j, \vartheta) \rangle_{w(\vartheta)} &:= \sum_{n,m} \frac{(-i)^n i^m}{n! m!} \int \langle 0 | \overline{T\Psi(x_1) \dots \Psi(x_n)} | a(\vartheta) \rangle \langle 0 | T\Psi(x_1) \dots \Psi(x_m) | b(\vartheta) \rangle \\ &\times \langle \varphi_0 | \partial(x_1) \dots \partial(x_n) \exp\{-\frac{1}{2} \int \partial(\xi) \overline{W_\vartheta(\xi \xi')} \partial(\xi') d\xi d\xi'\} \exp\{-\frac{1}{2} \int j(\xi) W_\vartheta(\xi, \xi') j(\xi') d\xi d\xi'\} \\ &\times j(x_1) \dots j(x_m) | \varphi_0 \rangle dx dx'. \end{aligned} \quad (\text{II.4})$$

For the evaluation of (II.4) we use the rest frame defined by

$$W_\vartheta(\xi, \xi') = F(\xi - \xi') + i_5 G_0 G f(\xi_0 - \vartheta) \delta(\xi - \xi') \quad (\text{II.5})$$

and introduce the new variables  $z = x - \vartheta e_0$ ,  $\zeta = \xi - \vartheta e_0$  for all occurring quantities. This gives  $W_\vartheta(\xi, \xi') = W_0(\zeta, \zeta')$  and

$$\begin{aligned} \langle \overline{\mathfrak{T}_a(j, \vartheta)} | \mathfrak{T}_b(j, \vartheta) \rangle_{w(\vartheta)} &= \sum_{n,m} \int \langle 0 | \overline{T\Psi(z_1 + \vartheta) \dots \Psi(z_n + \vartheta)} | a(\vartheta) \rangle \langle 0 | T\Psi(z + \vartheta) \dots \Psi(z + \vartheta) | b(\vartheta) \rangle \\ &\times \langle \varphi_0 | \partial(z_1 + \vartheta) \dots \partial(z_n + \vartheta) \exp\{-\frac{1}{2} \int \partial(\zeta + \vartheta) \overline{W_0(\zeta \zeta')} \partial(\zeta' + \vartheta) d\zeta d\zeta'\} \\ &\times \exp\{-\frac{1}{2} \int j(\zeta + \vartheta) W_0(\zeta \zeta') j(\zeta' + \vartheta) d\zeta d\zeta'\} j(z + \vartheta) \dots j(z + \vartheta) | \varphi_0 \rangle dz dz'. \end{aligned} \quad (\text{II.6})$$

Observing for the special case of the time translation

$$\Psi(z + \vartheta) = e^{-iH\vartheta} \Psi(z) e^{iH\vartheta}; \quad \langle 0 | e^{-iH\vartheta} = \langle 0 |; \quad e^{iH\vartheta} | a(\vartheta) \rangle = | a(0) \rangle \quad (\text{II.7})$$

and according to <sup>9</sup> for the sources

$$j(z + \vartheta) = V j(z) V^{-1}, \quad \partial(z + \vartheta) = V \partial(z) V^{-1}, \quad \langle 0 | V = \langle 0 |; \quad V^{-1} | 0 \rangle = | 0 \rangle \quad (\text{II.8})$$

we obtain from (II.6) by substitution of (II.7), (II.8)

$$\begin{aligned} \langle \mathfrak{T}_a(j, \vartheta) | \mathfrak{T}_b(j, \vartheta) \rangle_{w(\vartheta)} &= \sum_{n,m} \frac{(-i)^n i^m}{n! m!} \int \langle 0 | \overline{T\Psi(z_1) \dots \Psi(z_n)} | a(0) \rangle \langle 0 | T\Psi(z) \dots \Psi(z) | b(0) \rangle \\ &\times \langle \varphi_0 | \partial(z_1) \dots \partial(z_n) \exp\{-\frac{1}{2} \int \partial(\zeta) \overline{W_0(\zeta \zeta')} \partial(\zeta') d\zeta d\zeta'\} \\ &\times \exp\{-\frac{1}{2} \int j(\zeta) W_0(\zeta \zeta') j(\zeta') d\zeta d\zeta'\} j(z) \dots j(z) | \varphi_0 \rangle dz dz' = \langle \overline{\mathfrak{T}_a(j, 0)} | \mathfrak{T}_b(j, 0) \rangle_{w(0)}. \quad \text{q.e.d.} \end{aligned} \quad (\text{II.9})$$